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LETTER TO THE EDITOR

Multifractal amplitude fluctuations in the transfer-matrix approach to the statistics of disordered lattice models†

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Abstract. It is shown that the amplitude fluctuations of vector iterates in the transfer-matrix approach to the statistics of the one-dimensional random Ising chain correspond to an exact strange set characterised by a multifractal spectrum. The temperature and disorder dependence of the spectrum are also investigated.

There has been increasing interest recently in the characterisation of strange sets in terms of a universal distribution of fractal dimensions. Strange sets appear frequently (as strange attractors) in the theory of dynamical systems, and their multifractal nature is well documented (Benzi *et al* 1984, 1985, Jensen *et al* 1985, Halsey *et al* 1986a, Paladin and Vulpiani 1986a, b). The statistical description of fully developed turbulence and intermittency is another field of theoretical research that has benefited from the concept of multifractal sets of singularities present in the distribution of the velocity field (Frisch and Parisi 1983, Benzi *et al* 1984). Within the context of condensed matter theory, multifractal strange sets arise from the study of disordered systems. Examples are represented by the anomalous voltage distribution in random resistor networks (de Arcangelis *et al* 1985), the growth site distribution in diffusion limited aggregation (Turkevitch and Scher 1985, Halsey *et al* 1986b) and the anomalous scaling of the probability density at the localisation threshold in the Anderson model (Castellani and Peliti 1986, Paladin and Vulpiani 1986c).

The purpose of this letter is to point out the existence of an exactly solvable strange set (a two-scale Cantor set) in the amplitude fluctuations of the vector iterates of the transfer-matrix approach to the statistics of the random exchange Ising model. Disordered lattice models are central to the theoretical description of properties near phase transitions and critical points of fluid and magnetic mixtures. The transfer-matrix approach (Nightingale 1982) to disordered lattice statistics (see, e.g., Cheung and McMillan 1983) is a useful technique for extracting critical point properties, particularly for $d=2$ lattice space dimensions. However, a very large number of products of random matrices is usually involved in the calculations and this feature can lead in some cases (Bouchaud and Le Doussal 1986) to intermittency and sample dependence in the results of the iterations. Some of these features seem to be present also in the case of the disordered Ising model (Kaski 1981) and a characterisation in terms of a multifractal spectrum may help in understanding fluctuations in the transfer-matrix method.

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In McMillan's transfer-matrix approach to the disordered Ising model, one considers a d-dimensional bar of Ising spins, part of an hypercubic lattice, N sites in length and n sites in width. The partition function for a given configuration $\{J_{ij,i'j'}\}$ of nearest-neighbour bonds is

$$Z=\mathrm{Tr}\left[\prod_1^N T_j\right]=\left(\left|\left[\prod_1^N T_j\right]V_E\right|\right)(|V_E|)^{-1}=\exp(N\gamma_E) \qquad N\to\infty. \quad (1)$$

In the above equation, T_j is the transfer matrix relative to the sections j and $j+1$ of the bar,

$$T_j(\{S_{ij}\},\{S'_{ij+1}\})=\exp\left[\beta\sum_i\left(J_{ij,ij+1}S_{ij}S'_{ij+1}+\sum_\delta J_{ij,i+\delta j}S_{ij}S_{i+\delta j}\right)\right]$$

and the sums run over all sites i and nearest neighbours δ of the section. Also, γ_E is the largest Lyapunov exponent (LE) for the growth of an even vector V_E. Similarly, one can define the largest LE γ_0 for the growth of an odd vector V_0,

$$\left(\left|\left[\prod_1^N T_j\right]V_0\right|\right)(|V_0|)^{-1}=\exp(N\gamma_0) \qquad N\to\infty \quad (2)$$

so that the correlation length ξ of the bar is determined by

$$\xi=1/(\gamma_E-\gamma_0)$$

(Cheung and McMillan 1983). This formulation assumes an exponential decay of the correlation function at large distances, as well as the self-averaging of the free energy and correlation functions in the large N limit. The terms 'even' and 'odd' for an iteration vector refer to its canonical expansion in terms of Ising spin operators:

$$V(\{S\})=\sum_{\{m_i=0,1\}}a(m_1,m_2,\ldots,m_M)S_1^{m_1}S_2^{m_2}\ldots S_M^{m_M}$$

where $M=n^{d-1}$ is the total number of spins in the section. Accordingly, an even (odd) vector will satisfy $V(\{-S\})=V(\{S\})$, or $a(m_1,\ldots,m_M)=0$ for Σm_i odd ($V(\{-S\})=-V(\{S\})$, or $a(m_1,\ldots,m_M)=0$ for Σm_i even). The advantage of this representation is that transfer-matrix iterations can be carried out analytically by operating with the amplitudes $a(\{m_i\})$ and with their recursion relations. For the simplest case of a one-dimensional random exchange chain, these are

$$\begin{aligned}a'(0)&=2ca(0)\\ a'(1)&=2sa(1)\end{aligned} \quad (3)$$

with $c=\cosh(\beta J)$ and $s=\sinh(\beta J)$ and J a random bond variable with given distribution. The LE defined by equations (1) and (2) can then be extracted from the asymptotic behaviour of the iterated amplitudes:

$$a_0(\{m_i\})=\sum_\mu A_\mu(\{m_i\})$$

$$\lim_{N\to\infty}a_N(\{m_i\})=\sum_\mu\exp(N\gamma_\mu)A_\mu(\{m_i\})\to\exp(N\gamma)A(\{m_i\})$$

where Σ_μ runs over the 2^M eigenvectors of the limit matrix $\Gamma=\lim_{N\to\infty}(\Pi_1^N T_j)^{1/N}$, while γ and $A(\{m_i\})$ refer to the largest eigenvalue or LE. For N large but finite, the evaluation of γ_E and γ_0 is therefore affected by the fluctuations of the iterated amplitudes. Other formulations of the random transfer-matrix method will present similar features.

I will now show, for the case of the one-dimensional chain equation (3) with binary distribution $P(J) = p\delta(J - J_A) + (1-p)\delta(J - J_B)$, that these fluctuations are multifractal. In order to parametrise the fluctuations it is convenient, following Pietronero and Siebesma (1986), to associate a subinterval $[X_j, X_{j+1}] \subset [0, 1]$ with a given sequence of s iterations $\{J_1, J_2, \ldots, J_s\}$ and to associate the corresponding value of $a_s(m)$ with this interval. The set of subintervals $[X_j, X_{j+1}]$, with $j = 1, 2, \ldots, 2^s$, is constructed recursively by dividing $[0, 1]$ in two intervals $[0, p]$ and $[p, 1]$ at step $s = 1$, and subsequently by dividing each interval of step s into two intervals proportional to p and $1-p$ at step $s+1$. In analytic terms, a sequence $\{\sigma_j\} = \{J_1, J_2, \ldots, J_s\}$ of s iterations is associated with a number $X_j^{(s)}$ given by

$$X_j^{(s)} = \sum_{\nu=1}^{s} \Delta_\nu^{(j)} p^{\nu - k_{\nu-1}^{(j)}} (1-p)^{k_{\nu-1}^{(j)}}$$

$$k_s^{(j)} = \sum_{\nu=1}^{s} \Delta_\nu^{(j)}$$

where $\Delta_\nu = 0$ for $J_\nu = J_A$ and $\Delta_\nu = 1$ for $J_\nu = J_B$. For $s \to \infty$ a one to one correspondence between $[0, 1]$ and all possible configurations of the infinite chain is established. A normalised distribution $\tilde{a}_s(m, X)$ of the fluctuations is then constructed by introducing the average growth LE $\bar{\gamma}_E$ and $\bar{\gamma}_0$ for the even ($m = 0$) and odd ($m = 1$) amplitudes:

$$\langle a_N(0) \rangle = a_0(0) \exp(N\bar{\gamma}_E)$$

$$\langle a_N(1) \rangle = a_0(1) \exp(N\bar{\gamma}_0)$$

where $\langle \; \rangle$ denotes the average over all configurations of the $\{J_1, J_2, \ldots, J_N\}$ sequence of iterations. Owing to the rigorously log-binomial nature of the probability distribution, $\bar{\gamma}_E$ and $\bar{\gamma}_0$ are readily evaluated:

$$\bar{\gamma}_E = \ln[2(pc_A + (1-p)c_B)]$$

$$\bar{\gamma}_0 = \ln[2(ps_A + (1-p)s_B)]$$

with $c_{A,B} = \cosh(\beta J_{A,B})$, $s_{A,B} = \sinh(\beta J_{A,B})$. It should be noticed that use of these average growth LE does not lead to the correct formula for the correlation length of the chain (Wortis 1974),

$$\bar{\gamma}_E - \bar{\gamma}_0 \neq \xi^{-1} = -\ln[pt_A + (1-p)t_B] \qquad t_{A,B} = s_{A,B}/c_{A,B}.$$

The normalised distribution of amplitude fluctuations is then obtained as

$$\tilde{a}_s(0, X) = \exp(-s\bar{\gamma}_E) a_s(0, X)$$

$$\tilde{a}_s(1, X) = \exp(-s\bar{\gamma}_0) a_s(1, X)$$

so that $\langle \tilde{a}_s(m, X) \rangle = \int_0^1 \mathrm{d}X\, \tilde{a}_s(m, X) = a_0(m) = 1$. For a pure chain one has $\tilde{a}_N(m, X) = 1$; for a disordered chain, $\tilde{a}_N(m, X)$ is a map of the configurational fluctuations of the iterated amplitudes after N iterations. Figure 1 represents $\tilde{a}_8(0, X)$ for $p = 0.7$, $\beta J_A = 1$ and $\beta J_B = 0$. It is clear that at any step s of the iteration $\tilde{a}_s(0, X)$ takes $s+1$ distinct values

$$\{\tilde{a}_k^{(s)}\} = \{c_A^{s-k} c_B^k / [pc_A + (1-p)c_B]^s, k = 0, 1, \ldots, s\}$$

distributed with frequency $\binom{s}{k}$ over 2^s subintervals having $s+1$ distinct widths $\{l_k^{(s)}\} = \{p^{s-k}(1-p)^k\}$ (only even amplitude fluctuations are considered in detail here; for the odd amplitude, $s_{A,B}$ replaces $c_{A,B}$ and absolute values should be taken in the results).

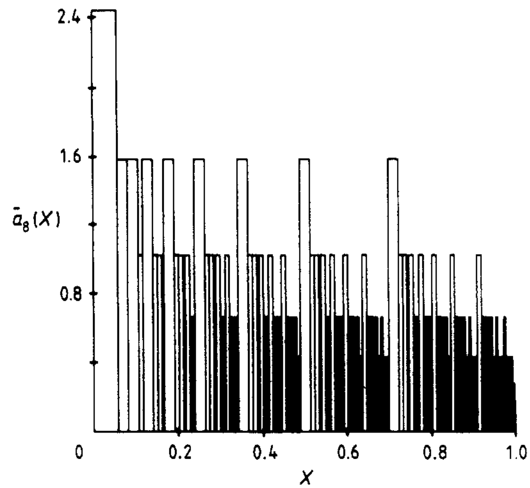

Figure 1. Normalised distribution of the fluctuations of the even amplitude after $N = 8$ iterations ($p = 0.7$, $\beta J_A = 1.0$, $\beta J_B = 0.0$).

The iterations of $\tilde{a}_s(0, X)$ are strictly self-similar and generate a two-scale Cantor set with basic scales $l_1 = p$ and $l_2 = 1 - p$ and measures

$$P_1 = pc_A/[pc_A + (1-p)c_B] \qquad P_2 = (1-p)c_B/[pc_A + (1-p)c_B].$$

After s iterations the measure of the strange set $\tilde{a}_s(m, X)$ is $P_k^{(s)} = \int_{l_k^{(s)}} \mathrm{d}X\, \tilde{a}_s(m, X) = l_k^{(s)} \tilde{a}_k^{(s)}$. It is now trivial to show that this strange set is exactly solvable (in the sense of Halsey *et al* (1986a)) by evaluating its partition function:

$$\Gamma_s(q, \tau) = \sum_{j=1}^{2^s} P_j^{(s)q}/l_j^{(s)\tau} = \sum_{k=0}^{s} \binom{s}{k} l_k^{(s)q-\tau} \tilde{a}_k^{(s)q} = [\Gamma_1(q, \tau)]^s$$

where $\Gamma_1(q, \tau)$ is the generator of this particular Cantor set's partition:

$$\Gamma_1(q, \tau) = P_1^q/l_1^\tau + P_2^q/l_2^\tau = [p^{q-\tau}c_A^q + (1-p)^{q-\tau}c_B^q]/[pc_A + (1-p)c_B]^q. \quad (4)$$

The multifractal distribution of singularities of the strange set $\tilde{a}_\infty(0, X)$ is now the Legendre transform of the function $\tau(q)$ defined by $\Gamma_1(q, \tau(q)) = 1$, or, by virtue of equation (4), by

$$\begin{aligned} &p^{q-\tau(q)}c_A^q + (1-p)^{q-\tau(q)}c_B^q = [pc_A + (1-p)c_B]^q \\ &\alpha = \mathrm{d}\tau(q)/\mathrm{d}q \qquad (5) \\ &f(\alpha) = \alpha q(\alpha) - \tau(q(\alpha)). \end{aligned}$$

As explained by Halsey *et al* (1986a), the function $f(\alpha)$ describes the strange set $\tilde{a}_\infty(m, X)$ as an interwoven family of singularities of type α, each distributed over a set embedded in $[0, 1]$ having fractal dimension $f(\alpha)$. It can be verified that for the pure chain (or for $\beta = 0$) $\alpha = 1$ and $f(\alpha) = 1$. For the disordered chain, $f(\alpha)$ generally assumes the characteristic convex shape having maximum value $f = d = 1$ (the Euclidean dimension of $[0, 1]$). $f(\alpha)$ can be determined for any values of the parameters p, βJ_A and βJ_B from the numerical solution of equation (5). For values of the parameters not too close to those characterising the pure limit, the simplified analytic treatment of Halsey *et al* (1986a) for the two-scale Cantor set can be used, which consists in solving

$$\alpha = [\ln P_1 + (r-1)\ln P_2]/[\ln l_1 + (r-1)\ln l_2]$$
$$f(\alpha) = [(r-1)\ln(r-1) - r\ln r]/[\ln l_1 + (r-1)\ln l_2]$$

by elimination of r, where P_1, P_2, l_1 and l_2 are the basic measures and length scales of the Cantor set. Figures 2 and 3 show the dependence of the multifractal spectrum of $\tilde{a}_\infty(0, X)$ on dilution and temperature, respectively, for the diluted Ising chain ($\beta J_B = 0$). It appears that the essential features of $f(\alpha)$ depend little on dilution and change more rapidly with temperature. In particular, amplitude fluctuations increase rapidly as the temperature is lowered, as expected, and as quantified by the spread of $f(\alpha)$. This implies that spin-glass and percolation (both $T = 0$) transition properties may be very difficult to study using the transfer-matrix technique.

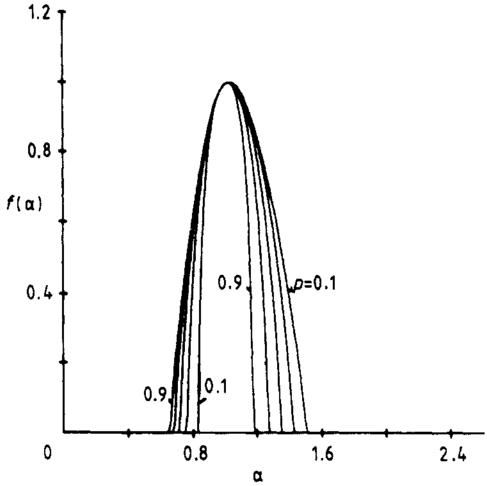

Figure 2. Dependence on dilution p for the spectrum $f(\alpha)$ of the fluctuations of the even amplitude ($\Delta p = 0.2$, $\beta J_A = 1.0$, $\beta J_B = 0.0$).

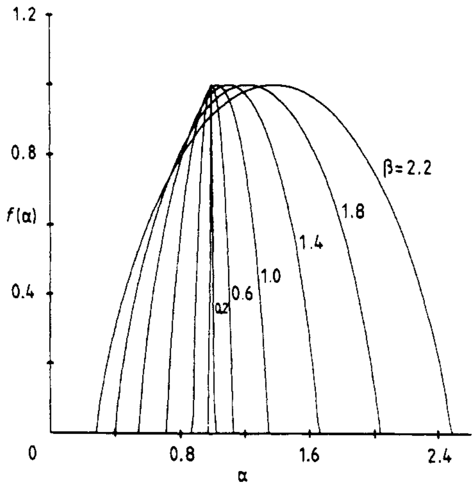

Figure 3. Dependence on temperature $1/\beta$ for the spectrum $f(\alpha)$ ($p = 0.5$, $J_A = 1.0$, $J_B = 0.0$).

Generalisation to an M-site section of the strip or bar of lattice sites is straightforward, by proper inclusion of the bonds within a section and of those in between sections in the specification of the 'one-dimensional' sequence of bonds $\{J_1, J_2, \ldots, J_s\}$. All the essential features presented for the one-dimensional chain are expected to hold. Given the proposed multifractal representation of the spatial intermittency of fully developed turbulence (Frisch and Parisi 1983), it is an intriguing question whether the multifractal amplitude fluctuations of the transfer-matrix approach to the disordered Ising model are in some way related to the apparent intermittency in the evaluation of the correlation length of strips by this method (Kaski 1981). Possibly, a more controlled study of the critical properties of random Ising strips and bars may be attained by employing the generalised set of exponents $\{\tau(q)\}$.

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